Semiclassical Methods in 2D QFT: Spectra and Finite–Size Effects

Valentina Riva

School of Mathematics, Institute for Advanced Study
Einstein Drive, Princeton, NJ 08540, USA
riva@ias.edu

Abstract

We review some recent results obtained in the analysis of two–dimensional quantum field theories by means of semiclassical techniques, which generalize methods introduced during the Seventies by Dashen, Hasllacher and Neveu and by Goldstone and Jackiw. The approach is best suited to deal with quantum field theories characterized by a non–linear interaction potential with different degenerate minima, that generates kink excitations of large mass in the small coupling regime. Under these circumstances, although the results obtained are based on a small coupling assumption, they are nevertheless non–perturbative, since the kink backgrounds around which the semiclassical expansion is performed are non–perturbative too. We will discuss the efficacy of the semiclassical method as a tool to control analytically spectrum and finite–size effects in these theories.

1 Introduction

Non–perturbative methods in quantum field theory (QFT) play a central rôle in theoretical physics, with applications in many areas, from string theory to condensed matter. During the last two decades, considerable progress has been registered in the study of two–dimensional systems, where exact results have been obtained in the particular situations of conformally invariant or integrable models (for detailed accounts of these achievements, see for instance Refs.[1], [2]).

A natural continuation of the above mentioned studies consists in developing some techniques to control analytically two–dimensional QFT which do not display conformal invariance or integrability and therefore are presently analyzed by perturbative or numerical methods only. This review summarizes the contributions of the author in this respect, obtained in collaboration with Giuseppe Mussardo and Galen Sotkov. Our main tool was an appropriate generalization and extension of semiclassical methods, which proved to be efficient in analysing non-perturbative effects in QFT since their introduction in the seminal works of Refs.[3, 4]. The semiclassical approach does not require integrability, therefore it can be applied on a large class of systems. At the same time, it permits to face problems which are not fully understood even in the integrable cases, such as the analytic study of QFT in finite volume. In particular, it has led to new non–perturbative results on form factors at a finite volume[5], spectra of non–integrable models[6] and energy levels of QFT on finite geometries[7, 8, 9]. Further achievements in understanding advantages and drawbacks of the semiclassical method have been presented by G. Mussardo in Ref.[10].

The semiclassical method is best suited to deal with those quantum field theories characterized by a non–linear interaction potential with different degenerate minima. These systems display kink excitations, associated to static classical backgrounds which interpolate between neighbouring minima, which generally have a large mass in the small coupling regime. Under these circumstances, although the results obtained are based on a small coupling assumption, they are nevertheless non-perturbative, since the kink backgrounds around which the semiclassical expansion is performed are non-perturbative too. The restriction on the variety of examinable theories imposed by the above requirements is rather mild, since non–linearity is the main feature of a wealth of relevant physical problems.

The review is organized as follows. After recalling the basic aspects of semiclassical quantization in Section 2, in Section 3 we discuss its application to the study of the spectrum in non–integrable QFT. Section 4 presents the analysis of finite–size effects, and we conclude in Section 5.

2 Semiclassical quantization

In this Section we will describe the two main tools used in the following to investigate non–integrable spectra and finite–size effects. The first is represented by the semiclassical quantization technique, introduced for relativistic field theories in a series of papers by Dashen, Hasslacher and Neveu (DHN)[3] by using an appropriate generalization of the WKB approximation in quantum mechanics. The second is a result due to Goldstone and Jackiw[4], which relates the form factors of the basic field between kink states to the Fourier transform of the classical solution describing the kink. For a complete review of these beautiful achievements, and complementary techniques developed by other groups during the Seventies, see Ref.[11]. Although semiclassical methods are naturally formulated for QFT in any dimension $d+1$, here we will only consider (1+1)–dimensional theories, in virtue of their simplified kinematics that allows for powerful applications of the semiclassical techniques.

2.1 DHN method

The semiclassical quantization of a field theory defined by a Lagrangian

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu\phi)\,(\partial^\mu\phi) - V(\phi) \tag{2.1}$$

is based on the identification of a classical background $\phi_{cl}(x,t)$ which satisfies the Euler–Lagrange equation of motion

$$\partial_\mu\partial^\mu\phi_{cl} + V'(\phi_{cl}) = 0\ . \tag{2.2}$$

The procedure is particularly simple and interesting if one considers finite–energy static classical solutions $\phi_{cl}(x)$ in 1+1 dimensions, usually called "kinks" or "solitons". They appear in field theories defined by a non–linear interaction $V(\phi)$ displaying discrete degenerate minima ϕ_i, which are constant solutions of the equation of motion and are called "vacua". The (anti)kinks interpolate between two next neighbouring minima of the potential, and they carry topological charges $Q = \pm 1$.

Being static solutions of the equation of motion, i.e. time independent in their rest frame, the kinks can be simply obtained by integrating the first order differential equation related to (2.2)

$$\frac{1}{2}\left(\frac{\partial\phi_{cl}}{\partial x}\right)^2 = V(\phi_{cl}) + A\ \ , \tag{2.3}$$

further imposing that $\phi_{cl}(x)$ reaches two different minima of the potential at $x \to \pm\infty$. These boundary conditions, which describe the infinite volume case, require the vanishing of the integration constant A. As we will see in the following, the kink solutions in a finite volume correspond instead to a non–zero value of A, related to the size of the system.

For definiteness in the illustration of the method, we will focus on the example of the ϕ^4 theory in the broken $\mathbb{Z}_2$ symmetry phase, defined by the potential

$$V(\phi) = \frac{\lambda}{4}\,\phi^4 - \frac{m^2}{2}\,\phi^2 + \frac{m^4}{4\lambda}\ . \tag{2.4}$$

This theory displays two degenerate minima at $\phi_\pm = \pm\frac{m}{\sqrt{\lambda}}$, and a static (anti)kink interpolating between them

$$\phi_{cl}(x) = (\pm)\,\frac{m}{\sqrt{\lambda}}\ \tanh\frac{mx}{\sqrt{2}}\ . \tag{2.5}$$

The corresponding classical energy, obtained by integrating the energy density $\varepsilon_{cl}(x) \equiv \frac{1}{2}\left(\frac{d\phi_{cl}}{dx}\right)^2 + V(\phi_{cl})$,

$$\mathcal{E}_{cl} \equiv \int_{-\infty}^{\infty} dx\, \varepsilon_{cl}(x) = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda} \;, \tag{2.6}$$

diverges as the interaction coupling $\lambda \to 0$, indicating that the solution is non–perturbative. Fig. 1 shows the potential, the classical kink and its energy density.

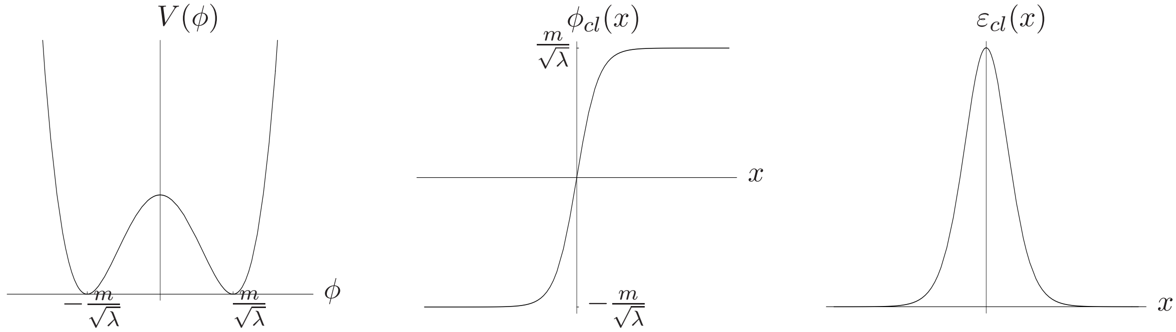

Figure 1: Potential $V(\phi)$, kink $\phi_{cl}(x)$ and energy density $\varepsilon_{cl}(x)$ in the broken ϕ^4 theory.

At quantum level, the kinks are localized and topologically stable excitations. An effective method for their semiclassical quantization has been developed by Dashen, Hasslacher and Neveu (DHN)[3] by using an appropriate generalization of the WKB approximation in quantum mechanics. The DHN method consists in splitting the field $\phi(x,t)$ into the static classical solution and its quantum fluctuations, i.e.

$$\phi(x,t) \;=\; \phi_{cl}(x) + \eta(x,t) \quad , \quad \eta(x,t) \;=\; \sum_k e^{i\omega_k t}\, \eta_k(x) \;,$$

and in further expanding the action of the theory in powers of η. This amount to an expansion in the interaction coupling λ, as for instance in the example (2.4)

$$\begin{aligned} \mathcal{S}(\phi) \;&=\; \int dx\, dt\, \mathcal{L}(\phi_{cl}) + \int dx\, dt\, \frac{1}{2}\,\eta(x,t)\left(\frac{d^2}{dt^2} - \frac{d^2}{dx^2} - m^2 + 3\lambda\phi_{cl}^2\right)\eta(x,t) + \\ &\quad + \lambda \int dx\, dt \left(\phi_{cl}\,\eta^3 + \frac{1}{4}\eta^4\right) \;. \end{aligned} \tag{2.7}$$

The semiclassical approximation consists in keeping only the quadratic terms in η. As a result of this procedure, $\eta_k(x)$ satisfies the so called “stability equation”

$$\left[-\frac{d^2}{dx^2} + V''(\phi_{cl})\right] \eta_k(x) \;=\; \omega_k^2\, \eta_k(x) \;, \tag{2.8}$$

together with certain boundary conditions. The semiclassical energy levels in each sector are then built in terms of the energy of the corresponding classical solution and the eigenvalues ω_i of the Scrödinger–like equation (2.8), i.e.

$$E_{\{n_i\}} \;=\; \mathcal{E}_{cl} + \hbar \sum_k \left(n_k + \frac{1}{2}\right)\omega_k + O(\hbar^2) \;, \tag{2.9}$$

where n_k are non–negative integers. In particular the ground state energy in each sector is obtained by choosing all $n_k = 0$ and it is therefore given by[1]

$$E_0 \,=\, \mathcal{E}_{cl} + \frac{\hbar}{2}\sum_k \omega_k + O(\hbar^2) \ . \tag{2.10}$$

This technique was applied in Ref.[3] to the kink background (2.5), in order to compute the first quantum corrections to its mass, whose leading order term is the classical energy. In this case, the stability equation (2.8) can be cast in hypergeometric form and therefore exactly solved. The semiclassical correction to the kink mass can be computed as the difference between the ground state energy in the kink sector and the one of the vacuum sector, plus a mass counterterm due to normal ordering (see Ref.[11] for details). The final result is

$$M \,=\, \frac{2\sqrt{2}}{3}\,\frac{m^3}{\lambda} + m\left(\frac{1}{6}\sqrt{\frac{3}{2}} - \frac{3}{\pi\sqrt{2}}\right) \ . \tag{2.11}$$

2.2 Classical solutions and form factors

A direct relation between the kink states and the corresponding classical solutions has been established by Goldstone and Jackiw[4], who have shown that the matrix element of the field ϕ between kink states is given, at leading order in the semiclassical expansion, by the Fourier transform of the kink background.

We will now derive this result in the example of the broken ϕ^4 theory (2.4), illustrating the assumptions behind it. Let us define the matrix element (also called form factor) of the basic field $\phi(x,t)$ between two one–kink states of momenta p_1 and p_2, as the Fourier transform of a function $\hat{f}(a)$, to be determined:

$$\langle\, p_2 \,|\, \phi(0) \,|\, p_1 \,\rangle \,=\, \int da\; e^{i(p_1-p_2)a}\; \hat{f}(a) \ . \tag{2.12}$$

Next consider the Heisenberg equation of motion for the quantum field $\phi(x,t)$

$$\left(\partial_t^2 - \partial_x^2\right)\phi(x,t) \,=\, m^2\,\phi(x,t) \,-\, \lambda\,\phi^3(x,t) \ , \tag{2.13}$$

and take the matrix elements of both sides[2]

$$[-(p_1-p_2)_\mu(p_1-p_2)^\mu]\, e^{-i(p_1-p_2)_\mu x^\mu}\langle\, p_2 \,|\, \phi(0) \,|\, p_1 \,\rangle \,=$$

$$=\, e^{-i(p_1-p_2)_\mu x^\mu}\,\left\{ m^2\,\langle\, p_2 \,|\, \phi(0) \,|\, p_1 \,\rangle \,-\, \lambda\,\langle\, p_2 \,|\, \phi^3(0) \,|\, p_1 \,\rangle \right\} \ . \tag{2.14}$$

We will now equate the two members of this equation at leading order in λ.

[1]From now on we will fix $\hbar = 1$, since the semiclassical expansion in $\hbar$ is equivalent to the expansion in the interaction coupling λ.

[2]Lorentz invariance imposes the relation $\langle\, p_2 \,|\, \phi(x,t) \,|\, p_1 \,\rangle \,=\, e^{-i(p_1-p_2)_\mu x^\mu}\langle\, p_2 \,|\, \phi(0) \,|\, p_1 \,\rangle$

In the left hand side of (2.14), the energy difference

$$(E_1 - E_2)^2 = \left(\frac{p_1^2 - p_2^2}{2M} + ...\right)^2 = O(\lambda^2)$$

can be neglected, since the kink momentum is very small compared to its mass, due to (2.6),(2.11). Hence the left hand side gives, at leading order,

$$\int da\; e^{i(p_1-p_2)a} \left(-\frac{d^2}{da^2}\hat{f}(a)\right) .$$

In the right hand side of (2.14), the cubic power $\langle\, p_2 \,|\, \phi^3(0) \,|\, p_1 \,\rangle$ can be expanded over a complete set of states with the same topological charge as the kink. These are given by one–kink states $|\, p \,\rangle$ and by kink + neutral states $|\, p, k_1, ..., k_m \,\rangle$, where k_i indicate the neutral states' momenta (the neutral states, also called "mesons", are the quantum excitations associated to constant classical backgrounds, i.e. to vacua). Our assumption is that, in the weak coupling limit, the corresponding matrix elements behave as $\langle\, p' \,|\, \phi(0) \,|\, p \,\rangle = O(1/\sqrt{\lambda})$ and $\langle\, p', k_1', ..., k_l' \,|\, \phi(0) \,|\, p, k_1, ..., k_m \,\rangle = O(\lambda^{(l+m-1)/2})$. This assumption, which will find confirmation *a posteriori*, relies on the fact that the kink classical background is of order $1/\sqrt{\lambda}$ itself, and that the emission or absorption of every meson carry a factor $\sqrt{\lambda}$ [3]. In virtue this assumption, the leading term is obtained when the intermediate states are all one–kink states:

$$-\lambda \sum_{p,q} \langle\, p_2 \,|\, \phi(0) \,|\, p \,\rangle \langle\, p \,|\, \phi(0) \,|\, q \,\rangle \langle\, q \,|\, \phi(0) \,|\, p_1 \,\rangle \;=\; -\lambda \int da\; e^{i(p_1-p_2)a}\; [\, \hat{f}(a) \,]^3 .$$

Hence, at leading order in λ, the function $\hat{f}(a)$ obeys the same differential equation satisfied by the kink solution, i.e.

$$\frac{d^2}{da^2}\hat{f}(a) = \lambda[\, \hat{f}(a) \,]^3 - m^2 \hat{f}(a) . \tag{2.15}$$

This means that we can take $\hat{f}(a) = \phi_{cl}(a)$, adjusting its boundary conditions by an appropriate choice for the value of the constant A in eq. (2.3).

Therefore, we finally obtain

$$\langle\, p_2 \,|\, \phi(0) \,|\, p_1 \,\rangle \;=\; \int da\; e^{i(p_1-p_2)a}\; \phi_{cl}(a) \;+\; \text{higher order terms} . \tag{2.16}$$

Along the same lines, it is easy to prove that the form factor of an operator expressible as a function of ϕ is given by the Fourier transform of the same function of ϕ_{cl}. For instance, the form factor of the energy density operator ε can be computed performing the Fourier transform of $\varepsilon_{cl}(x) = \frac{1}{2}\left(\frac{d\phi_{cl}}{dx}\right)^2 + V(\phi_{cl})$.

Similar arguments lead to semiclassical expression for the matrix elements between excited states of the kink, or states containing kink and mesons (see Ref.[4],[11]).

[3] This can be intuitively understood by noticing that, in the expansion (2.7) of the interaction $V(\phi)$, the leading perturbative term is of order $\lambda\phi_{cl}$, i.e. of order $\sqrt{\lambda}$.

3 Non–integrable quantum field theories

As we mentioned in the Introduction, integrable QFT in $(1+1)$ dimensions admit a non–perturbative treatment which led to exact results and relevant predictions in statistical mechanics and condensed matter applications. Nonetheless, both theoretical reasons and applications call for a deeper understanding of non–integrable systems as well. In general, these are analyzed through perturbative or numerical techniques only, and some of their basic data, such as the mass spectrum, are often not easily available. There are, however, two favorable situations when analytical tools can be used to extract non–perturbative results.

The first case is that of non–integrable theories which can be seen as small perturbations of integrable ones. An approach called Form Factor Perturbation Theory (FFPT)[12] has been developed, which exploits the non–perturbative knowledge of the integrable theory in order to get quantitative predictions on mass spectrum, scattering amplitudes and other physical quantities in these systems.

A complementary situation is represented by theories having kink excitations of large mass in their semiclassical limit. In this case, the semiclassical method introduced in Section 2 is a natural candidate to obtain analytic non–perturbative results. In this Section we focus on this approach, and we apply it to the analysis of the spectrum in some non–integrable theories. Our main tool will be a generalization of the result by Goldstone and Jackiw discussed in Sect. 2.2.

3.1 Relativistic formulation of Goldstone and Jackiw's result

In order to apply Goldstone and Jackiw's result (2.16) to the study of the spectrum in non–integrable QFT, we first need to refine it in order to overcome its drawback of being expressed in terms of the difference of space momenta of the two kinks. The original formulation is not Lorentz covariant, and the antisymmetry under the interchange of momenta makes problematic any attempt to go in the crossed channel and obtain the matrix element between the vacuum and a kink–antikink state.

In order to overcome these problems, in Ref.[5] we have refined the result by using, instead of the space–momenta of the kinks, their rapidity variable θ, defined in terms of energy and momentum as

$$E \equiv M \cosh\theta \;, \qquad p \equiv M \sinh\theta \;. \tag{3.17}$$

This parameterization is particularly convenient, since the rapidity difference is a Lorentz invariant of a two–particles scattering process, as can be seen from its relation with the Mandelstan variable s:

$$s = (p_1+p_2)_\mu (p_1+p_2)^\mu = m_1^2 + m_2^2 + 2 m_1 m_2 \cosh(\theta_1-\theta_2) \;. \tag{3.18}$$

The approximation of large kink mass used by Goldstone and Jackiw can be realized considering the rapidity as very small. For example, in the ϕ^4 theory (2.4), where the kink mass M is of order $1/\lambda$, we work under the hypothesis that θ is of order λ. In this way we get $E \simeq M$, $p \simeq M\,\theta \ll M$. It is easy to see that the proof of (2.16) outlined in Sect. 2.2 still holds, if we define the form factor between kink states as the Fourier transform with respect to the rapidity difference $\theta = \theta_1 - \theta_2$

$$\langle p_1 | \phi(0) | p_2 \rangle \equiv f(\theta) \equiv M \int da \, e^{i M\theta a} \phi_{cl}(a) \; . \tag{3.19}$$

The use of rapidity variable permits to analytically continue the form factor (3.19) to the crossed channel, via the transformation $\theta \to i\pi - \theta$, which is equivalent to the transformation from the Mandelstam variable s to t. We then have

$$F_{K\bar{K}}(\theta) \equiv \langle 0 | \phi(0) | p_1, \bar{p}_2 \rangle = f(i\pi - \theta) \; . \tag{3.20}$$

A first use of the matrix elements (3.20) is to estimate the leading behaviour in λ of the spectral representation of correlation functions in a regime of momenta dictated by the assumption of small kink rapidity. Here we will discuss their second main application, which permits to extract information about the spectrum of the theory. The two–particle form factors share the same s–channel dynamical poles of the scattering matrix, which correspond to the creation of kink–antikink bound states. Their behaviour in the vicinity of a singularity is

$$F_{K\bar{K}}(\theta) \sim \frac{i\,\Gamma^b_{K\bar{K}}}{(\theta - \theta^*)} \, F_b(\theta^*) \; ,$$

where $\Gamma^b_{K\bar{K}}$ is the on–shell three–particle coupling constant between kink, antikink and the bound state b, and the poles are located at $\theta^* = i(\pi - u)$, with $0 < u < \pi$. The process is pictorially represented in Fig. 2. Since the corresponding singularity in the s–variable is of the form $(s - m_b^2)^{-1}$, it follows from (3.18) that the mass of the bound state can be expressed as

$$m_b^2 = m_K^2 + m_{\bar{K}}^2 + 2 m_K m_{\bar{K}} \cos u = \left(2M \sin \frac{u}{2} \right)^2 \; . \tag{3.21}$$

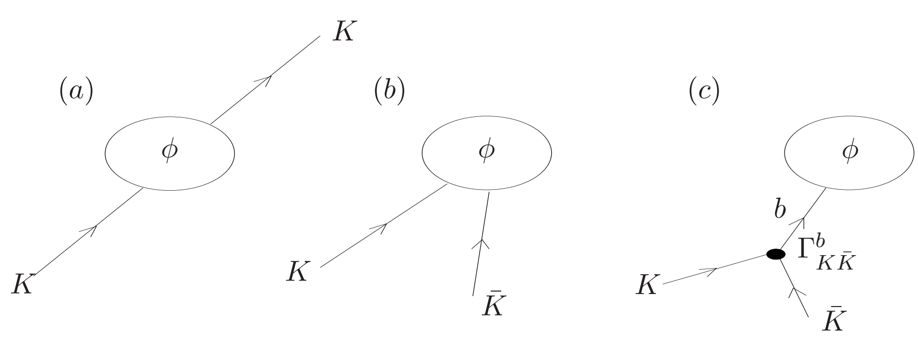

Figure 2: Pictorial representation of (a) the form factor $f(\theta)$, (b) the crossed channel form factor $F_2(\theta)$, (c) the form factor $F_2(\theta^*)$ at the dynamical pole θ^*.

It is worth noticing that this procedure for extracting the semiclassical bound states masses is remarkably simpler than the standard DHN method of quantizing the corresponding classical backgrounds, because in general these solutions depend also on time and have a much more complicated structure than the kink ones. Moreover, in non–integrable theories these backgrounds could even not exist as exact solutions of the field equations: this happens for example in the ϕ^4 theory, where the DHN quantization has been performed on some approximate backgrounds[3].

3.2 Broken ϕ^4 theory

Let us now apply the semiclassical method to the analysis of the spectrum in the ϕ^4 field theory in the $\mathbb{Z}_2$ broken symmetry phase. This non–integrable theory, defined by the potential (2.4), is invariably referred to as a paradigm for a wealth of physical phenomena. In spite of this deep interest, however, its non–perturbative features are still poorly understood.

The main properties of the potential (2.4) and its kink background (2.5) have been already discussed in Sect. 2.1. The form factor (3.19) takes the form

$$\langle\, p_2 |\, \phi(0) |\, p_1 \,\rangle \;=\; \frac{4}{3} i\pi \left(\frac{m}{\sqrt{\lambda}}\right)^3 \frac{1}{\sinh\left(\frac{2}{3}\pi \frac{m^2}{\lambda}\,\theta\right)} \,, \tag{3.22}$$

where the kink mass is approximated at leading order by the classical energy $M = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda}$. The dynamical poles of $F_{K\bar{K}}(\theta)$ are located at

$$\theta_n = i\pi \left[1 - \frac{3}{2\pi}\,\frac{\lambda}{m^2}\,n\right] \,, \qquad\qquad 0 < n < \frac{2\pi}{3}\frac{m^2}{\lambda} \,, \tag{3.23}$$

and the corresponding bound states masses are given by

$$m_b^{(n)} = 2M \sin\left[\frac{3}{4}\,\frac{\lambda}{m^2}\,n\right] = n\,\sqrt{2}\,m \left[1 - \frac{3}{32}\,\frac{\lambda^2}{m^4}\,n^2 + ...\right] . \tag{3.24}$$

Note that the leading term is consistently given by multiples of $\sqrt{2}m$, which is the known mass of the elementary boson of the theory[4]. This spectrum exactly coincides with the one derived in Ref.[3] by building approximate time–dependent classical solutions to represent the neutral excitations. Since $m_b^{(3)} > 2m_b^{(1)}$, even when more than two particles are allowed by the value of λ in (3.23), only the first two are stable, while the others are resonances (see Ref.[10] for further comments and generalizations).

3.3 Efficacy and limitations of the semiclassical method

We have seen in the previous Section that the semiclassical method is an efficient tool to get the spectrum of the ϕ^4 theory. The natural question to be addressed now is how reliable the method is to study other types of potential.

First, let us mention an example where the semiclassical results can be compared, in the appropriate regime of couplings, with exact results obtained in virtue of integrability[13]. This is the case of the Sine–Gordon model, defined by the potential

$$V(\phi) \;=\; \frac{m^2}{\beta^2}\,(1 - \cos\beta\phi) \tag{3.25}$$

(see the first picture in Fig. 3). For this model, the semiclassical results are in very good agreement with exact ones also for values of β which extend beyond the semiclassical limit (see

[4] The elementary bosons represent the excitations over the vacua, i.e. the constant backgrounds $\phi_\pm = \pm\frac{m}{\sqrt{\lambda}}$, therefore their square mass is given by $V''(\phi_\pm) = 2\,m^2$.

Refs.[7],[10] for details). The spectrum consists of soliton and antisoliton excitations, which classically interpolate between two neighbouring minima of (3.25), and a tower of neutral states, called "breathers", associated to every minima.

A very interesting non–integrable theory is defined by the Double Sine–Gordon potential

$$V(\phi) \;=\; \frac{m^2}{\beta^2}\,(1-\cos\beta\phi) \;+\; \frac{\lambda}{\beta^2}\,\cos\left(\frac{\beta}{2}\phi+\delta\right) \;+\; const\;, \tag{3.26}$$

(see Fig. 3) which has several applications in statistical mechanics and condensed matter physics.

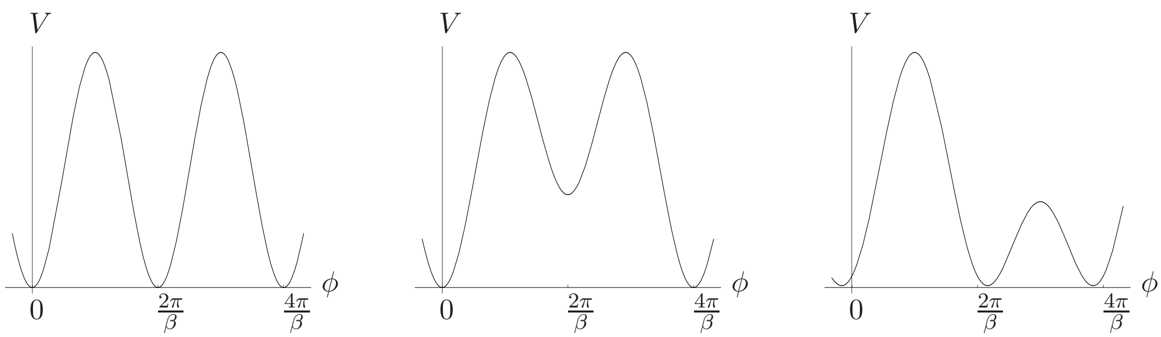

Figure 3: Potential $V(\phi)$ defined in (3.25), in (3.26) with $\delta = 0$ and in (3.26) with $\delta = \frac{\pi}{2}$, respectively.

From the theoretical point of view, this model is an ideal laboratory where to compare results obtained by FFPT around the integrable Sine–Gordon model[14, 15] (for small λ) and semiclassical methods[6] (for small β). Depending on the value of the phase δ, the potential (3.26) displays different qualitative features, which can be grouped in two classes: $\delta \neq \frac{\pi}{2}$ and $\delta = \frac{\pi}{2}$. We will now briefly summarize the results obtained in Ref.[6], to which we refer the reader for details.

Let us first describe the $\delta \neq \frac{\pi}{2}$ case, by focusing on the particular value $\delta = 0$ (the qualitative features are the same for all other values). As shown in Fig. 3, as soon as a $\lambda \neq 0$ in switched on in the potential, the degeneracy between neighbouring vacua of the Sine–Gordon model is spoiled, with a consequent confinement of the solitons. At the same time, a new static kink appears, which interpolates between the vacua at $\phi = 0$ and $\phi = \frac{4\pi}{\beta}$ in Fig. 3. The semiclassical method can be applied to this excitation to determine the corresponding neutral bound states. Complementarily, FFPT is capable of estimating the corrections to the masses of Sine–Gordon breathers which, being non–topological excitations, are not confined as the corresponding solitons. Therefore, the two techniques combine together in the analysis of the full spectrum of the model.

A qualitatively different scenario appears at the value $\delta = \frac{\pi}{2}$. In this case, the degeneracy of two neighbouring minima of the Sine–Gordon model is not destroyed, hence the solitons are not confined but simply deformed into a "large kink" and a "small kink" which interpolate between the minima separated by a larger or smaller wall, respectively (see Fig. 3). In this case, a straightforward application of the semiclassical method leads to wrong results. It seems that there are two towers of neutral states with different masses around each vacuum, one obtained as bound states of long kink and long antikink, and the other from short kink and short antikink. This is in contradiction with the fact that the breathers of the unperturbed Sine–Gordon model are non degenerate, and it is moreover disproved by numerical analysis[16]. The controversy has been clarified in Ref.[10], by noticing that, at leading order in the coupling in which the semiclassical form factor (3.19) is computed, the short and long kink are invisible to each other, while the correct spectrum of neutral states must be the result of the interaction between the two different kinks. An exact formula for the masses of neutral states is still unknown, yet the semiclassical method provides useful information for the limiting cases when the masses of long and short kink are very close, or when the large kink is much heavier than the small (see Ref.[10] for details).

The problem outlined above is typical of every potential where kinks of different masses originate from the same vacuum. Therefore, particular care has to be adopted in applying the semiclassical method to those cases. For further comments and developments, see the discussion in Ref.[10].

4 Finite–size effects

Quantum field theory on a finite size is a subject of both theoretical and practical interest. Besides providing a way to control the extrapolation procedure of numerical simulations, the study of a theory in finite volume also permits to follow the renormalization group flow between the ultraviolet (UV) conformal limit and the infrared (IR) massive behaviour. Clearly understood in CFT (see Ref.[17]), finite–size effects can be tackled non–perturbatively in integrable QFT as well, with the so–called Thermodynamic Bethe Ansatz method[18], which is a combination of analytical and numerical procedures.

In this Section, we will discuss the contributes to this subject obtained through the semiclassical method[5, 7, 8, 9]. Once the proper classical solutions are identified to describe a given geometry, the spectral function in finite volume can be easily estimated by adapting the Goldstone and Jackiw's result. Furthermore, the finite–volume kinks can be quantized semiclassically via the DHN technique, which permits to write in analytic form the discrete energy levels as functions of the size of the system.

We will now explicitly show the construction in the case of the Sine–Gordon model, defined by the potential (3.25)

$$V(\phi) \,=\, \frac{m^2}{\beta^2}\,(1-\cos\beta\phi)\;.$$

We chose this integrable theory as the guiding example in this Section, since the analysis of its finite size effects is technically simpler than in the ϕ^4 theory. Full details about the ϕ^4 model can be found in Refs.[5, 9]. Being an integrable theory, the Sine–Gordon model has been already studied on a finite size by appropriate extensions of the Thermodynamic Bethe ansatz[19]. In comparison with those techniques, the semiclassical method provides more explicit anlytical results. It would be interesting to perform a quantitative comparison between the two approaches, in order to directly control the range of validity of the semiclassical approximation, as it was done in the infinite volume case.

4.1 Classical solutions and form factors

The basic ingredient in the semiclassical study of finite size effects is the classical kink solution on a finite volume. This can be obtained by solving eq. (2.3) with an appropriate constant A to encode the chosen boundary conditions. We will now focus on a cylindrical geometry of circumference R, where the b.c. for a single kink can be quasi–periodic or antiperiodic:

$$\phi(x+R) \;=\; \frac{2\pi}{\beta} \pm \phi(x)\;, \tag{4.27}$$

and correspond to $A>0$ and $-2\frac{\beta^2}{m^2}<A<0$ in eq. (2.3), respectively. The associated classical solutions are expressed in terms of elliptic functions[5], whose modulus k is related to the size R:

$$\phi^{+}_{cl}(x) \;=\; \tfrac{\pi}{\beta}+\tfrac{2}{\beta}\,\mathrm{am}\left(\tfrac{mx}{k},k^2\right)\;, \qquad k^2=\frac{2}{2+\frac{m^2}{\beta^2}A}\;, \quad R=\frac{2}{m}\,k\,\mathbf{K}(k^2) \tag{4.28}$$

$$\phi^{-}_{cl}(x) \;=\; \tfrac{2}{\beta}\,\arccos\left[\,k\,\mathrm{sn}\left(mx,k^2\right)\right]\;, \qquad k^2=\frac{\frac{m^2}{\beta^2}A+2}{2}\;, \quad R=\frac{2}{m}\,\mathbf{K}(k^2) \tag{4.29}$$

(see Fig. 4).

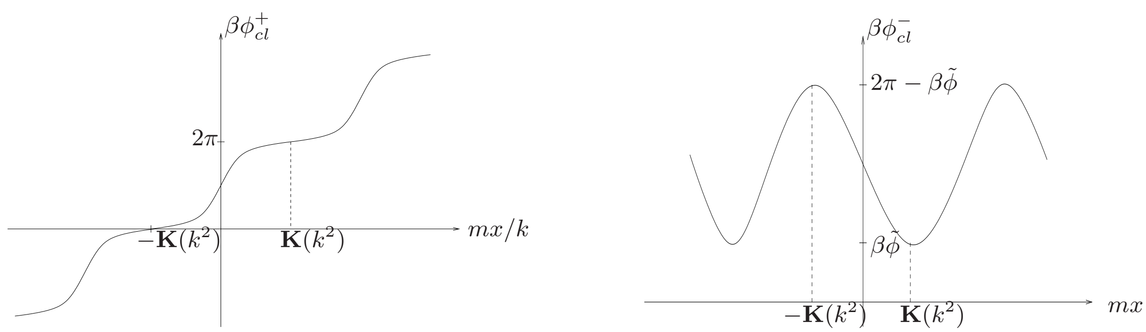

Figure 4: Solutions of eq. (2.3), $A>0$ (left hand side), $-2<A<0$ (right hand side).

For simplicity, we will now only discuss the quasi–periodic case ϕ^+_{cl}, which is characterized by a classical energy

$$\mathcal{E}_{cl}(R) \;=\; \frac{8m}{\beta^2}\left[\frac{\mathbf{E}(k^2)}{k}+\frac{k}{2}\left(1-\frac{1}{k^2}\right)\mathbf{K}(k^2)\right]\;. \tag{4.30}$$

A complete treatment of the antiperiodic background ϕ^-_{cl} can be found in Ref.[5, 9], while the study of other backgrounds related to a strip geometry with Dirichlet boundary conditions has been performed in Ref.[8].

We will first show that the relation between classical solutions and semiclassical form factors presented in Section 2.2 also holds in finite volume. This comes from the possibility of choosing

[5]For definitions and properties of elliptic integrals and Jacobi elliptic functions, see Ref.[20]

$\hat{f}(a)$ as a solution of eq. (2.3) with any constant A, which is related to the size of the system. We have now to consider the matrix elements of $\phi(0)$ between two eigenstates $|p_{n_1}\rangle$ and $|p_{n_2}\rangle$ of the finite volume hamiltonian H_R. These states can be naturally labelled with the so-called "quasi-momentum" variable p_n, which corresponds to the eigenvalues of the translation operator on the cylinder (multiples of π/R), and appears in the space dependent part of eq. (2.14) in the case of finite volume[6]. Defining θ_n as the "quasi-rapidity" of the kink states by

$$p_n = M(R)\sinh\theta_n \simeq M(R)\theta_n \ ,$$

we can now write the form factor at a finite volume by replacing the Fourier integral transform with a Fourier series expansion:

$$f(\theta_n) \, = \, \langle p_{n_2}|\,\phi(0)\,|p_{n_1}\rangle \, = \, M(R)\int\limits_{-R/2}^{R/2} da\, e^{i\,M(R)\theta_n a}\phi_{cl}(a) \ , \tag{4.31}$$

where

$$M(R)\,\theta_n \, \simeq \, p_{n_1} - p_{n_2} \, = \, \frac{(2n_1-1)\pi}{R} \, - \, \frac{(2n_2-1)\pi}{R} \, \equiv \, \frac{2n\pi}{R} \ .$$

This result, of very general applicability, adds to previous studies of finite volume form factors[21], which on the contrary deeply rely on the integrable structure of the considered models. In our particular example, the Fourier transform of (4.28) gives

$$\begin{aligned} f(\theta_n) \quad &= \quad \frac{2\pi}{\beta}\left\{\frac{M}{2}\,R\,\delta_{M\theta_n,0} - i\,\frac{1-\delta_{M\theta_n,0}}{\theta_n}\left[\cos\left(M\theta_n\,R/2\right) - \frac{\sin\left(M\theta_n\,R/2\right)}{M\theta_n R/2}\right] + \right. \\ &\qquad \left. + \, i\,\frac{1}{\theta_n\,\cosh\left(k\,\mathbf{K}'\frac{M}{m}\,\theta_n\right)}\right\} \ , \end{aligned}$$

where the kink mass M can be approximated at leading order with its classical energy (4.30).

4.2 Energy levels

We will now briefly sketch the semiclassical derivation of energy levels in finite volume (for a detailed discussion see Ref.[7]). The aim is to obtain analytical expressions for the energies $E_i(R)$ as functions of the circumference R. The procedure consists in adapting the DHN quantization, outlined in Sect. 2, to the finite geometry.

We already know the explicit expression of the classical kink (4.28) satisfying quasi–periodic boundary conditions. In order to construct the scaling functions, we have to solve the corresponding Schrödinger equation (2.8) and to derive an analytical expression for its frequencies ω_k. Here we will not discuss the mathematical details of this procedure, which is explained in Ref.[7]; in essence, the stability equation turns out to be of the so–called $N = 1$ Lamé type, for

[6] For large R, the quasi-momentum is related to the free momentum p^{∞} of the infinite volume asymptotic states by the so–called Bethe ansatz equation $p_n^{\infty} + \frac{\delta(p_n^{\infty})}{R} = \frac{2n\pi}{R} \equiv p_n$, where $\delta(p^{\infty})$ is a phase shift which encodes the information about the interaction.

which exact solutions are known in terms of elliptic and Weierstrass functions. The final result for the frequencies is

$$\omega_n^2 = \frac{m^2}{k^2} \left[\frac{2-k^2}{3} - \mathcal{P}(iy_n) \right] , \tag{4.32}$$

where y_n is defined by

$$2\mathbf{K} i \, \zeta(iy_n) + 2y_n \zeta(\mathbf{K}) = 2n\pi , \tag{4.33}$$

which has the physical meaning of momentum quantization (see Ref.[20] for definitions of the Weierstrass functions $\mathcal{P}$ and ζ). By inverting the relation between R and k in (4.28), it is easy to plot the frequencies (4.32) (see Fig. 5), which represent the energies of the excited states (2.9) with respect to the ground state (2.10).

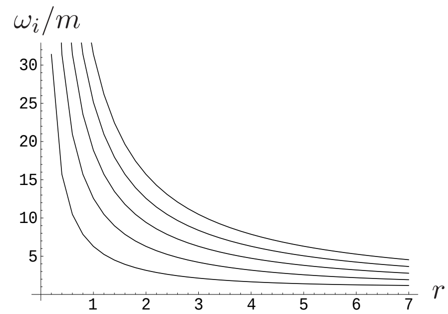

Figure 5: The first few levels defined in (4.32)

In order to derive the ground state energy in the kink sector, it is necessary to regularize the infinite sum (2.10), by subtracting to it the ground state energy in the vacuum sector and an appropriate mass counterterm. The procedure can be explicitly carried out in the ultraviolet (UV) and infrared (IR) regimes, when $r = mR \to 0$ and $r \to \infty$, respectively. It is particularly interesting to compare the corresponding limiting behaviours with asymptotic results already known for the Sine–Gordon theory.

The small r expansion is

$$\frac{E_0^{\text{kink}}(R) - E_0^{\text{vac}}(R)}{m} = \frac{2\pi}{r} \frac{\pi}{\beta^2} + \frac{1}{\beta^2} r - \frac{1}{8} \left(\frac{r}{2\pi} \right)^2 - \left(\frac{r}{2\pi} \right)^3 \left[\frac{1}{8} \zeta(3) - \frac{1}{4} (2 \log 2 - 1) - \frac{\pi}{2\beta^2} \right] + \dots , \tag{4.34}$$

and it has to be compared with the Conformal Field Theory prediction[17]

$$E_0(R) = \frac{2\pi}{R} \left(\Delta_0 - \frac{c}{12} \right) + BR + \dots , \tag{4.35}$$

where c is the central charge, Δ_0 is the lowest scaling dimension in the sector under consideration, and B is the so–called bulk coefficient. For the Sine–Gordon model the bulk energy term is given by[22]

$$B = 16 \frac{m^2}{\gamma^2} \tan \frac{\gamma}{16} , \qquad \text{with} \quad \gamma = \frac{\beta^2}{1 - \frac{\beta^2}{8\pi}} ,$$

hence the corresponding term in (4.34) has the correct small–β behaviour. Moreover, the scaling dimension of the kink–creating operator is known to be[23]

$$\Delta_0 \,=\, \frac{\pi}{\beta^2}\ ,$$

again in agreement with (4.34)[7].

We should now look at the IR limit of the kink energy, and compare it with the asymptotic approach to the infinite volume kink mass predicted by Lüscher's theory[24]:

$$M(R) - M(\infty) \,=\, 2\,m\,\sin\left(\frac{\pi}{2}+\frac{\gamma}{16}\right)\,\cot\frac{\gamma}{16}\,e^{-m\sin\left(\frac{\pi}{2}+\frac{\gamma}{16}\right)R} \,+\, O(e^{-2mR})\ .$$

At leading order in β, this behaviour can be already detected at the level of the classical energy (4.30), whose IR expansion is

$$\mathcal{E}_{cl}(R) \,=\, \frac{8m}{\beta^2} \,+\, m\,\frac{32}{\beta^2}\,e^{-mR} \,+\, O(e^{-2mR})\ ,$$

where the kink mass in infinite volume is $M_\infty = \frac{8m}{\gamma}$.

The successful check with known UV and IR asymptotic behaviours confirms the ability of the semiclassical method to analytically describe the scaling functions of SG model in the one–kink sector.

5 Conclusions

We have briefly reviewed some fruitful applications of semiclassical methods to the study of non–integrable QFT and finite–size effects in two dimensions.

In order to keep the discussion reasonably short, we have omitted to mention some interesting phenomena which can be captured by the semiclassical method, like unstable resonance states and false vacuum decay. Details can be found in the original literature.

Let us mention two of the several open problems which deserve further attention. First, the study of semiclassical form factors at higher order in the coupling constant would lead not only to quantitatively better results, but also to a satisfactory understanding of the spectrum in models where two different kinks emerge from the same vacuum, as we discussed in the Double Sine–Gordon case. Second, a systematic investigation of the finite–size spectrum beyond the one–kink sector is still missing. This requires to find appropriate time–dependent classical solutions on a finite size and apply to them the semiclassical quantization procedure.

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[7]The central charge c does not appear in (4.34), since it cancels out in the subtraction of the ground state energy in the vacuum sector

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